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# The classical basis for the $\kappa$ -Poincaré Hopf algebra and doubly special relativity theories

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## Abstract

Several issues concerning the quantum  $\kappa$ -Poincaré algebra are discussed and reconsidered here. We propose two different formulations of the  $\kappa$ -Poincaré quantum algebra. Firstly we present a complete Hopf algebra formula of  $\kappa$ -Poincaré in classical Poincaré basis. Further by adding one extra generator, which modifies the classical structure of the Poincaré algebra, we eliminate nonpolynomial functions in the  $\kappa$ -parameter. Hilbert space representations of such algebras make doubly special relativity (DSR) similar to Stueckelberg's version of (proper-time) relativistic quantum mechanics.

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## 1. Introduction

The concept of a quantum group was introduced more than 20 years ago in [1–4] (see also [5, 6]), and since then the subject has been widely investigated under different approaches and has gained popularity and all sorts of applications (see, e.g. [7–16]). One of the applications is to consider the notion of a quantum group as noncommutative generalization of a symmetry group of the physical system, which means that the quantum group takes the place of the symmetry group of spacetime, i.e. the Poincaré group. Roughly speaking, quantum groups are the deformations of some classical structures as groups or Lie algebras, which are made in the category of Hopf algebras. Similarly, quantum spaces are noncommutative generalizations (deformations) of ordinary spaces. The most important in physics and mathematically the simplest one seem to be the canonical and the Lie-algebraic quantum deformations. It has not taken long to note that in the description of the short-distance structure of spacetime (at the Planck scale) the existing symmetries may be modified including deformation of Poincaré symmetry. Moreover, it has been suggested that the symmetries of the  $\kappa$ -deformed Minkowski space should be described in terms of the Hopf algebra [7–11]. The studies on this

type of deformations were inspired by [7], where the quantum  $\kappa$ -Poincaré algebra with the masslike deformation parameter was first proposed. The algebraic structure of the  $\kappa$ -Poincaré algebra has gained even more attention and since then has been intensively analyzed from the mathematical and physical point of view. For historical reviews see, e.g. [14–16] and the references therein.

A chance for the physical application of this theory appeared when an extension of special relativity was proposed in [17, 18], and another one, showing a different point of view, in [19]. This extension includes two observer-independent scales, the velocity of light and the scale of mass, now called ‘doubly special relativity’ (DSR). Also, various phenomenological aspects of DSR theories have been studied in, e.g. [20]. For comparison of these two approaches see, e.g. [21]. The connection between  $\kappa$ -deformation and DSR theory in first formulation (DSR1) has been shown (see, e.g. [18, 22, 23]) including the conclusion that the spacetime of DSR must be noncommutative as a result of the Hopf structure of this algebra.

The  $\kappa$ -Poincaré algebra, as well as DSR, has been studied extensively and has found many applications besides physics at the Planck scale gravity, also in elementary particle physics and quantum field theory (see, e.g. [17–25] and references therein). The  $\kappa$ -Poincaré Hopf algebra has been discovered in the so-called standard basis [7] inherited from the anti-de Sitter basis by the contraction procedure. For this basis only the rotational sector remains algebraically undeformed. Introducing bicrossproduct basis allows us to leave the Lorentzian generators undeformed. This basis is the easier form of the  $\kappa$ -Poincaré algebra basis and was postulated in [10, 11]. In this form, the Lorentz subalgebra of the  $\kappa$ -Poincaré algebra, generated by rotations and boosts, is not deformed and the difference is only in the way the boosts act on momenta. There is also a change in the co-algebraic sector; the coproducts are no longer trivial, which has the already mentioned consequence: the spacetime of DSR is noncommutative.

It is well known that the Drinfeld–Jimbo quantization algorithm relies on simultaneous deformations of the algebraic and co-algebraic sectors and it is applicable to semisimple Lie algebras [1, 2]. In particular, this implies the existence of classical basis for Drinfeld–Jimbo quantized algebras. Strictly speaking, the Drinfeld–Jimbo technique cannot be applied to the Poincaré non-semisimple algebra which has been obtained by the contraction procedure from the Drinfeld–Jimbo deformation of the anti-deSitter (simple) Lie algebra  $\mathfrak{so}(3, 2)$ . Nevertheless, the  $\kappa$ -Poincaré quantum group shares many properties of the original Drinfeld–Jimbo quantization. These include existence of the classical basis, the square of the antipode and the solution to the specialization problem (see section 3).

In this paper we define the  $\kappa$ -Poincaré (Hopf) algebra in its classical Poincaré Lie algebra basis. The constructions of such basis were previously investigated in several papers [26, 27]. Particularly, the explicit formulas expressing classical basis in terms of the bicrossproduct have been obtained therein. Explicit formulas for coproducts can be found in different (realization dependent) context in [22, 28], see also [14]. To the best of our knowledge, there are no other examples of Drinfeld–Jimbo-type deformation expressed in a classical Lie-algebraic basis.

It is known that different DSR models are defined by different choices of basis in the universal envelop of the Poincaré Lie algebra. They lead to different  $\kappa$ -Poincaré coproducts. Now we are in a position to demonstrate that when these models are compared in the same classical basis, they differ by different operator realizations in the space of scalar-valued functions on a spacetime manifold. This result in some sense allows us to distinguish between the description of DSR1 and DSR2 theories. For the special choice of realization, we recover a well-known bicrossproduct form of the  $\kappa$ -Poincaré algebra and the standard DSR model. Moreover, according to the formalism developed in our previous paper [16], we have a wide range of models and deformed dispersion relations related to them at our disposal (section 4).

## 2. $\kappa$ -Poincaré Hopf algebra in classical basis

We shall use a standard so-called physical basis  $(\mathcal{M}_k, \mathcal{N}_k, \mathcal{P}_\mu)$  of the Poincaré Lie algebra  $\mathfrak{P}^{1,3}$  consisting of the Lorentz subalgebra  $\mathfrak{L}^{1,3}$  of rotation  $\mathcal{M}_i$  and boost  $\mathcal{N}_i$  generators:

$$[\mathcal{M}_i, \mathcal{M}_j] = \iota \epsilon_{ijk} \mathcal{M}_k, \quad [\mathcal{M}_i, \mathcal{N}_j] = \iota \epsilon_{ijk} \mathcal{N}_k, \quad [\mathcal{N}_i, \mathcal{N}_j] = -\iota \epsilon_{ijk} \mathcal{M}_k \quad (1)$$

supplemented by Abelian four-momenta  $\mathcal{P}_\mu = (\mathcal{P}_0, \mathcal{P}_k)$  ( $\mu = 0, \dots, 3, k = 1, 2, 3$ ) with the following commutation relations:

$$\begin{aligned} [\mathcal{M}_j, \mathcal{P}_k] &= \iota \epsilon_{jkl} \mathcal{P}_l, & [\mathcal{M}_j, \mathcal{P}_0] &= 0, \\ [\mathcal{N}_j, \mathcal{P}_k] &= -\iota \delta_{jk} \mathcal{P}_0, & [\mathcal{N}_j, \mathcal{P}_0] &= -\iota \mathcal{P}_j. \end{aligned} \quad (2)$$

We take the Lorentzian metric  $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$  for rising and lowering indices.

The algebra  $(\mathcal{M}_k, \mathcal{N}_k, \mathcal{P}_\mu)$  can be extended in the standard way to a Hopf algebra by defining on the universal enveloping algebra  $U_{\mathfrak{P}^{1,3}}$  the coproduct  $\Delta_0$ , the counit  $\epsilon$  and the antipode  $S_0$ , where the nondeformed-primitive coproduct, the antipode and the counit are given:

$$\Delta_0(X) = X \otimes 1 + 1 \otimes X, \quad S_0(X) = -X, \quad \epsilon(X) = 0 \quad (3)$$

for  $X \in \mathfrak{P}^{1,3}$ . In addition  $\Delta_0(1) = 1 \otimes 1, S_0(1) = 1$  and  $\epsilon(1) = 1$ . For the purpose of deformation one has to extend further this Hopf algebra by considering formal power series in  $\kappa^{-1}$ , and correspondingly considering the Hopf algebra  $(U_{\mathfrak{P}^{1,3}}[[\kappa^{-1}]], \cdot, \Delta_0, S_0, \epsilon)$  as a topological Hopf algebra with the so-called h-adic topology [5, 6]. Quantum deformations of this Hopf algebra are controlled by classical  $r$ -matrices satisfying the classical Yang–Baxter (YB) equation: homogeneous or inhomogeneous. The relation between the classical  $r$ -matrix  $\mathfrak{r}$  and a universal (quantum)  $r$ -matrix  $\mathcal{R}$  reads

$$\mathcal{R} = 1 + \frac{1}{\kappa} \mathfrak{r} \quad \text{mod} \left( \frac{1}{\kappa^2} \right), \quad (4)$$

where  $\frac{1}{\kappa}$  denotes the deformation parameter. In the case of  $r$ -matrices satisfying homogeneous YB equations the co-algebraic sector is twist-deformed while algebraic one remains classical [1]. Additionally, one can also apply existing twist tensors to related Hopf module-algebras in order to obtain quantized, e.g. spacetimes (see [14–16]). In contrast, Drinfeld–Jimbo quantization, corresponding to inhomogeneous  $r$ -matrices, relies on suitable deformation of the algebraic and co-algebraic sectors simultaneously. Therefore, the classification of quantum deformations is done by means of classification of the corresponding classical  $r$ -matrices: homogeneous and inhomogeneous one.

In the case of relativistic symmetries, such classification (complete for the Lorentz and almost complete for Poincaré algebras) has been performed in [29] (see also [30] where this classification scheme has been extended). Particularly, the  $r$ -matrix which corresponds to  $\kappa$ -deformation of the Poincaré algebra is given by

$$r = N_i \wedge P^i \quad (5)$$

and it satisfies the inhomogeneous (modified) Yang–Baxter equation:

$$[[r, r]] = M_{\mu\nu} \wedge P^\mu \wedge P^\nu. \quad (6)$$

Therefore, one does not expect to obtain the  $\kappa$ -Poincaré coproduct by twist. However, most of the items on that list contain homogeneous  $r$ -matrices. Explicit twists for them have been provided in [31] (for superization see [32, 33]); the corresponding quantization has been carried out in [13].

Our purpose in this paper is to formulate the  $\kappa$ -Poincaré Hopf algebra in a classical Poincaré basis. We would like to mention that the complete treatment of this problem was not considered before (see [26, 27]). One defines the deformed (quantized) coproducts  $\Delta_\kappa$  and the antipodes  $S_\kappa$  on  $\mathcal{U} \equiv U_{\mathfrak{q}^{1,3}}[[\kappa^{-1}]]$  leaving algebraic sector classical (untouched) like in the case of twisted deformation:

$$\Delta_\kappa(\mathcal{M}_i) = \Delta_0(\mathcal{M}_i) \tag{7}$$

$$\Delta_\kappa(\mathcal{N}_i) = \mathcal{N}_i \otimes 1 + \Pi_0^{-1} \otimes \mathcal{N}_i - \frac{1}{\kappa} \epsilon_{ijm} \mathcal{P}_j \Pi_0^{-1} \otimes \mathcal{M}_m \tag{8}$$

$$\Delta_\kappa(\mathcal{P}_i) = \mathcal{P}_i \otimes \Pi_0 + 1 \otimes \mathcal{P}_i \tag{9}$$

$$\Delta_\kappa(\mathcal{P}_0) = \mathcal{P}_0 \otimes \Pi_0 + \Pi_0^{-1} \otimes \mathcal{P}_0 + \frac{1}{\kappa} \mathcal{P}_m \Pi_0^{-1} \otimes \mathcal{P}^m \tag{10}$$

and the antipodes

$$S_\kappa(\mathcal{M}_i) = -\mathcal{M}_i, \quad S_\kappa(\mathcal{N}_i) = -\Pi_0 \mathcal{N}_i - \frac{1}{\kappa} \epsilon_{ijm} \mathcal{P}_j \mathcal{M}_m \tag{11}$$

$$S_\kappa(\mathcal{P}_i) = -\mathcal{P}_i \Pi_0^{-1}, \quad S_\kappa(\mathcal{P}_0) = -\mathcal{P}_0 + \frac{1}{\kappa} \vec{\mathcal{P}}^2 \Pi_0^{-1} \tag{12}$$

where

$$\Pi_0 \doteq \frac{1}{\kappa} \mathcal{P}_0 + \sqrt{1 - \frac{1}{\kappa^2} \mathcal{P}^2} \quad \text{and} \quad \Pi_0^{-1} \doteq \frac{\sqrt{1 - \frac{1}{\kappa^2} \mathcal{P}^2} - \frac{1}{\kappa} \mathcal{P}_0}{1 - \frac{1}{\kappa^2} \vec{\mathcal{P}}^2} \tag{13}$$

are just shortcuts;  $\mathcal{P}^2 \doteq \mathcal{P}_\mu \mathcal{P}^\mu \equiv \vec{\mathcal{P}}^2 - \mathcal{P}_0^2$ , and  $\vec{\mathcal{P}}^2 = \mathcal{P}_i \mathcal{P}^i$ . Let us stress the point that the above expressions are formal power series in the parameter  $\frac{1}{\kappa}$ , e.g.

$$\sqrt{1 - \frac{1}{\kappa^2} \mathcal{P}^2} = \sum_{n \geq 0} \frac{(-1)^n}{\kappa^{2n}} \binom{0.5}{n} [\mathcal{P}^2]^n \tag{14}$$

where  $\binom{0.5}{n} = \frac{0.5(0.5-1)\dots(0.5-n+1)}{n!}$  are binomial coefficients. From the above one calculates

$$\Delta_\kappa(\Pi_0) = \Pi_0 \otimes \Pi_0, \quad \Delta_\kappa(\Pi_0^{-1}) = \Pi_0^{-1} \otimes \Pi_0^{-1}, \quad S_\kappa(\Pi_0) = \Pi_0^{-1} \tag{15}$$

as well as

$$\Delta_\kappa\left(\sqrt{1 - \frac{1}{\kappa^2} \mathcal{P}^2}\right) = \sqrt{1 - \frac{1}{\kappa^2} \mathcal{P}^2} \otimes \Pi_0 - \frac{1}{\kappa} \Pi_0^{-1} \otimes \mathcal{P}_0 - \frac{1}{\kappa^2} \mathcal{P}_m \Pi_0^{-1} \otimes \mathcal{P}^m. \tag{16}$$

To complete the definition one leaves the counit  $\epsilon$  undeformed. Let us observe that  $\epsilon(\Pi_0) = \epsilon(\Pi_0^{-1}) = 1$ . It is also worth noticing that the square of the antipode (11)–(12) is given by a similarity transformation<sup>3</sup>, i.e.

$$S_\kappa^2(X) = \Pi_0 X \Pi_0^{-1}.$$

Substituting now

$$P_0 \doteq \kappa \ln \Pi_0, \quad P_i \doteq \mathcal{P}_i \Pi_0^{-1} \quad \Rightarrow \quad \Pi_0 = e^{\frac{P_0}{\kappa}} \tag{17}$$

one gets the deformed coproducts of the form

$$\Delta_\kappa(P_0) = 1 \otimes P_0 + P_0 \otimes 1, \quad \Delta_\kappa(P_k) = e^{-\frac{P_0}{\kappa}} \otimes P_k + P_k \otimes 1 \tag{18}$$

<sup>3</sup> In the case of twisted deformation the antipode itself is given by the similarity transformation.

$$\Delta_\kappa (\mathcal{N}_i) = \mathcal{N}_i \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes \mathcal{N}_i - \frac{1}{\kappa} \epsilon_{ijm} P_j \otimes \mathcal{N}_m. \tag{19}$$

Similarly the commutators of new generators can be obtained as

$$[\mathcal{N}_i, P_j] = -\frac{i}{2} \delta_{ij} \left( \kappa \left( 1 - e^{-\frac{2P_0}{\kappa}} \right) + \frac{1}{\kappa} \vec{P}^2 \right) + \frac{i}{\kappa} P_i P_j \tag{20}$$

with the remaining one being the same as for the Poincaré Lie algebra (1)–(2). This proves that our deformed Hopf algebra (1)–(2), (7)–(10) is Hopf isomorphic to the  $\kappa$ -Poincaré Hopf algebra [7] written in its bicrossproduct basis  $(\mathcal{M}_i, \mathcal{N}_i, P_\mu)$  [10]. From now on we shall denote the Hopf algebra  $U_{\mathfrak{P}^{1,3}}[[\kappa^{-1}]]$  (with  $\kappa$ -deformed coproduct) by  $\mathcal{U}(1, 3)[[\kappa^{-1}]]$ .

The following immediate comments are now in order.

- (i) Substituting  $\mathcal{N}_i = M_{0i}$  and  $\epsilon_{ijk} \mathcal{M}_k = M_{ij}$  the above result easily generalizes to the case of the  $\kappa$ -Poincaré Hopf algebra in an arbitrary spacetime dimension  $n$  (with the Lorentzian signature).
- (ii) Although  $\mathfrak{P}^{1,n-1}$  is the Lie subalgebra of  $\mathfrak{P}^{1,n}$  the corresponding Hopf algebra  $\mathcal{U}(1, n-1)[[\kappa^{-1}]]$  (with  $\kappa$ -deformed coproduct) is not the Hopf subalgebra of  $\mathcal{U}(1, n)[[\kappa^{-1}]]$ .
- (iii) Changing the generators by a similarity transformation  $X \rightarrow SXS^{-1}$  for  $X \in (\mathcal{M}_i, \mathcal{N}_i, P_\mu)$  leaves the algebraic sector (1)–(2) unchanged but in general it changes coproducts (7)–(10). Here  $S$  is assumed to be an invertible element in  $\mathcal{U}(1, 3)[[\kappa^{-1}]]$ . Both commutators and coproducts (7)–(10) are preserved provided that  $S$  is group-like, i.e.  $\Delta_\kappa(S) = S \otimes S$ , e.g.  $\Pi_0$ . For physical applications it might also be useful to consider other (nonlinear) changes of basis, e.g. in the translational sector. Therefore, the algebra  $\mathcal{U}(1, 3)[[\kappa^{-1}]]$  is a convenient playground for developing Magueijo–Smolin-type DSR theories [19, 34] (DSR2) even if we do not intend to take into account coproducts. But the coproducts are there and can be used, e.g. in order to introduce an additional law for four-momenta. In this situation the  $\kappa$ -deformed coproducts are not necessarily the privileged one and the additional law can be determined by, e.g. the twisted coproducts [13]. However,  $\kappa$ -deformed coproducts are consistent with  $\kappa$ -Minkowski commutation relations and give the  $\kappa$ -Minkowski spacetime module algebra structure [14, 16, 37].

### 3. New algebraic form of the $\kappa$ -Poincaré Hopf algebra

The mathematical formalism of quantum groups requires us to deal with formal power series. Therefore, the parameter  $\kappa$  has to stay formal, i.e. undetermined. Particularly, we cannot assign any particular numerical value to it and consequently any fundamental constant of nature, like, e.g. the Planck mass cannot be related to it. There are in principle two methods to remedy this situation and allow  $\kappa$  to admit a constant value.

The first one is to reformulate algebra in such a way that all infinite series will be eliminated on the abstract level. In the traditional Drinfeld–Jimbo approach this is always possible by using the so-called specialization method or  $q$ -deformation (see e.g. [5, 6]).<sup>4</sup> The idea is to replace  $\mathcal{P}_0$  by two group-like elements  $\Pi_0, \Pi_0^{-1}$  and ‘forget’ relation (13). This provides, for any specific (complex) numerical value  $\kappa \neq 0$ , a new quantum algebra  $\mathcal{U}_\kappa(1, 3)$ . It is defined as a universal, unital associative algebra generated by eleven generators  $(\mathcal{M}_i, \mathcal{N}_i, \mathcal{P}_i, \Pi_0, \Pi_0^{-1})$  being a subject of the standard Poincaré Lie-algebra commutation relations (1)–(2) except those containing  $\mathcal{P}_0$ . These last should be replaced by the following new ones ( $\Pi_0$  and  $\Pi_0^{-1}$  are considered mutually inverse):

<sup>4</sup> In some physically motivated papers a phrase ‘ $q$ -deformation’ is considered as an equivalent of the Drinfeld–Jimbo deformation. In this section we shall, following general terminology of [5, 6], distinguish between ‘ $h$ -adic’ and ‘ $q$ -analog’ Drinfeld–Jimbo deformations since they are not isomorphic.

$$[\mathcal{P}_i, \Pi_0] = [\mathcal{M}_j, \Pi_0] = 0, \quad [\mathcal{N}_i, \Pi_0] = -\frac{i}{\kappa} \mathcal{P}_i \tag{21}$$

$$[\mathcal{N}_i, \mathcal{P}_j] = -\frac{i}{2} \delta_{ij} \left( \kappa (\Pi_0 - \Pi_0^{-1}) + \frac{1}{\kappa} \vec{p}^2 \Pi_0^{-1} \right). \tag{22}$$

The Hopf algebra structure is determined on  $\mathcal{U}_\kappa(1, 3)$  by the same formulas (7)–(15) except those containing  $\mathcal{P}_0$ . It should be noted that these formulas contain only finite powers of the numerical parameter  $\kappa$ . The generator  $\mathcal{P}_0$  can now be introduced as

$$\mathcal{P}_0 \equiv \mathcal{P}_0(\kappa) \doteq \frac{\kappa}{2} \left( \Pi_0 - \Pi_0^{-1} \left( 1 - \frac{1}{\kappa^2} \vec{p}^2 \right) \right). \tag{23}$$

Thus, a subalgebra generated by the elements  $(\mathcal{M}_i, \mathcal{N}_i, \mathcal{P}_i, \mathcal{P}_0)$  is, of course, isomorphic to the universal envelope of the Poincaré Lie algebra, i.e.  $U_{\mathfrak{p}^{1,3}} \subset \mathcal{U}_\kappa(1, 3)$ . But this is not a Hopf subalgebra. Therefore, the original (classical) Casimir element  $\mathcal{C} \equiv -\mathcal{P}^2 = \mathcal{P}_0^2 - \vec{p}^2$  has, in terms of the generators  $(\Pi_0, \Pi_0^{-1}, \vec{P})$ , a rather complicated form. We can adopt at our disposal a simpler (central) element instead:

$$\mathcal{C}_\kappa \doteq \kappa^2 (\Pi_0 + \Pi_0^{-1} - 2) - \vec{p}^2 \Pi_0^{-1}, \tag{24}$$

which one may make responsible for deformed dispersion relations [16]. For comparison see, e.g. [35]. Both elements are related by

$$\mathcal{C} = \mathcal{C}_\kappa \left( 1 + \frac{1}{4\kappa^2} \mathcal{C}_\kappa \right) \quad \text{and} \quad \sqrt{1 + \frac{1}{\kappa^2} \mathcal{C}} = 1 + \frac{1}{2\kappa^2} \mathcal{C}_\kappa. \tag{25}$$

Finally, one should note that Hopf algebras  $\mathcal{U}_\kappa(1, 3)$  are isomorphic Hopf algebras for different values of  $\kappa$ . This is so since rescaling  $\mathcal{P}_i \mapsto \frac{1}{\kappa} \mathcal{P}_i$  makes  $\mathcal{U}_\kappa(1, 3) \cong \mathcal{U}_1(1, 3)$ .

#### 4. Hilbert space realizations

The second method allowing us to specify a value of  $\kappa$  relies on representation theory. Let us consider a representation of the Poincaré Lie algebra in a Hilbert space  $\mathfrak{h}$ . This leads to embedding of the entire enveloping algebra  $U_{\mathfrak{p}^{1,3}}$  into the space  $\mathcal{L}(\mathfrak{h})$  of linear operators over  $\mathfrak{h}$ . Thus some elements from  $\mathcal{U}(1, 3)[[\kappa^{-1}]]$  after substituting certain numerical value for  $\kappa$  can be considered as operators acting on  $\mathfrak{h}$ . Roughly speaking specialization appears via the spectral theorem on the level of Hilbert space realization. Thus, in fact, one deals with a representation of  $\mathcal{U}_\kappa(1, 3)$  instead of  $\mathcal{U}(1, 3)[[\kappa^{-1}]]$ . As an illustrative example one may consider a Stueckelberg’s proper-time Hilbert space of square integrable complex-valued (wave) functions on  $\mathbb{R}^4$ , i.e.  $\mathfrak{h} = L^2(\mathbb{R}^4, d^4x)$  [36] (see [11] for different representation). There are canonical commutation relations between (local) momentum and position operators

$$[p_\mu, x^\nu] = -i\delta_\mu^\nu, \quad [p_\mu, p_\nu] = [x_\mu, x_\nu] = 0 \tag{26}$$

represented by standard multiplication and differentiation operators:  $x^\mu$  and  $p_\mu = -i\partial_\mu$ . The representation of the Poincaré Lie algebra in this Hilbert space can be chosen, for example, as

$$\mathcal{M}_i = \frac{1}{2} \epsilon_{ijm} (x_j p_m - x_m p_j), \quad \mathcal{N}_i = \frac{\kappa}{2} x_i \left( e^{-2\frac{p_0}{\kappa}} - 1 \right) + x_0 p_i - x_i \Delta + \frac{1}{\kappa} x^k p_k p_i \tag{27}$$

$$\mathcal{P}_i = p_i e^{\frac{p_0}{\kappa}}, \quad \mathcal{P}_0 = \kappa \sinh \left( \frac{p_0}{\kappa} \right) + \frac{1}{2\kappa} \vec{p}^2 e^{\frac{p_0}{\kappa}} \tag{28}$$

where  $\Delta = -\vec{p}^2$  denotes the Laplace operator. Now all operators in the above formulas are well defined for a constant value of  $\kappa$  as Hilbert space operators. Moreover, it turns out that the

operators  $(\mathcal{M}_i, \mathcal{N}_i, p_\mu)$  constitute the bicrossproduct basis. Therefore, dispersion relations expressed in canonical momenta  $p_\mu$  are the standard DSR:

$$\mathcal{C}_\kappa = \kappa^2 \left( e^{-\frac{1}{2} \frac{p_0}{\kappa}} - e^{\frac{1}{2} \frac{p_0}{\kappa}} \right)^2 + \Delta e^{\frac{p_0}{\kappa}} \tag{29}$$

$$m_0^2 = \left[ 2\kappa \sinh \left( \frac{p_0}{2\kappa} \right) \right]^2 - \vec{p}^2 e^{\frac{p_0}{\kappa}}. \tag{30}$$

One can note that boost generators (27) in this representation are not Hermitian, because of the last term. However, the Hermitian representation of the  $\kappa$ -Poincaré algebra can be determined as

$$\mathcal{M}_i = \frac{1}{2} \epsilon_{ijm} (x_j p_m - x_m p_j), \quad \mathcal{N}_i = \frac{\kappa}{2} x_i \left( e^{-\frac{p_0}{\kappa}} - e^{\frac{p_0}{\kappa}} \right) + x_0 p_i e^{-\frac{p_0}{\kappa}} - x_i \Delta e^{-\frac{p_0}{\kappa}} \tag{31}$$

$$\mathcal{P}_i = p_i, \quad \mathcal{P}_0 = \kappa \sinh \left( \frac{p_0}{\kappa} \right) + \frac{1}{2\kappa} \vec{p}^2 e^{-\frac{p_0}{\kappa}} \tag{32}$$

and the dispersion relation is

$$m_0^2 = \left[ 2\kappa \sinh \left( \frac{p_0}{2\kappa} \right) \right]^2 - \vec{p}^2 e^{-\frac{p_0}{\kappa}}. \tag{33}$$

In both cases above, the representation of the element  $\Pi_0$  in the Hilbert space realization is given by the same formula  $\Pi_0 = e^{\frac{p_0}{\kappa}}$  (cf also (17)) (see [11, 12]). In the minimal case, connected with the Weyl–Poincaré algebra, in physical  $n = 4$  dimensions [16] the representation of the Poincaré algebra  $(M_i, N_i, \mathcal{P}_\mu)$  reads

$$M_i = -\frac{i}{2} \epsilon_{ijm} (x_j \partial_m - x_m \partial_j) \tag{34}$$

$$N_i = -x_i \left[ \frac{p_0}{2} \left( 2 + \frac{p_0}{\kappa} \right) + \Delta \right] \left( 1 + \frac{p_0}{\kappa} \right)^{-1} - i x_0 \partial_i. \tag{35}$$

The generator  $\Pi_0$  has now the form  $\Pi_0 = 1 + \frac{p_0}{\kappa}$  and the deformed Casimir operator

$$\mathcal{C}_\kappa = \frac{p_0^2 - \vec{p}^2}{1 + \frac{p_0}{\kappa}} \tag{36}$$

leads to following dispersion relation:

$$m_0^2 \left( 1 + \frac{p_0}{\kappa} \right) = p_0^2 - \vec{p}^2, \tag{37}$$

which is not deformed for (free) massless particles. We close this paper with an open question concerning the choice of a ‘physical’ Casimir operator leading to the correct dispersion relation and its operator realization.

### 5. Conclusions

In this paper we have introduced two different Hopf algebras of  $\kappa$ -Poincaré as quantum deformation of the Drinfeld–Jimbo type. The first one is related to ‘h-adic’ topology which forces the parameter  $\kappa$  to stay abstract and undetermined. All formulas for coproducts have been written intrinsically in a classical Lie algebra basis which is very typical for the twisted Drinfeld deformation technique. As it has already been explained, in the introduction, the



existence of such a classical basis for the Drinfeld–Jimbo deformations is a direct consequence of their formalism. However, the explicit construction is a highly nontrivial mathematical problem and to the best of our knowledge it was investigated mainly for the case of  $\kappa$ -Poincaré [26, 27]. Particularly, the formulas expressing classical basis in terms of the bicrossproduct one have been obtained therein.

The second definition relies on reformulating the Hopf algebra structure in such a way that infinite series disappear: it provides the one-parameter family of mutually isomorphic Hopf algebras labeled by a numerical (complex in general) parameter  $\kappa$ . So, the particular value of  $\kappa$  becomes irrelevant. From the physical point of view, one is allowed to work in the system of natural (Planck) units with  $\hbar = G = c = 1$  without changing the mathematical properties of the underlying quantum model. In this way the so-called specialization problem for the deformation parameter  $\kappa$  has been solved.<sup>5</sup> Finally, it has been shown that different (proper-time) Hilbert space representation of this algebra can be understood as the one corresponding to different DSR-type models providing different dispersion relations. Therefore, we believe that our research might also be helpful to distinguish between two approaches to doubly special relativity theories.

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<sup>5</sup> Similar problem in the bicrossproduct basis has been previously studied in [38].

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